Finite Element Formulation for Dynamics of Moving Plates

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1 Introduction

Plates are flat structures with one dimension much smaller than the other two and are widely used in modeling structures like aircraft wings. A fully intrinsic formulation, i.e. devoid of displacement and rotation variables, for the dynamics of a moving composite plate has been presented by Hodges *et al.* (2009). A variable-order finite element technique is presented and applied to beams by Patil and Hodges (2011). In this paper, the idea from the finite element paper is used to develop a solution methodology for the dynamics of moving plate.

2 Nonlinear, Intrinsic Beam Equations

The nonlinear, fully intrinsic governing equations for the dynamics of a moving plate are given as

$$\begin{aligned}
&N_{11,1} + (N_{12} + \mathfrak{N}), 2 - K_{13}(N_{12} - \mathfrak{N}) - K_{23}N_{22} + Q_1K_{11} + Q_2K_{21} + f_1 = \dot{P}_1 + \Omega_1 P_3 - \Omega_3 P_2 \\
&N_{22,1} + (N_{12} + \mathfrak{N}), 1 - K_{23}(N_{12} - \mathfrak{N}) - K_{13}N_{11} + Q_1K_{12} + Q_2K_{22} + f_2 = \dot{P}_2 + \Omega_3 P_1 - \Omega_2 P_3 \\
&Q_{1,1} + Q_{2,2} - K_{11}N_{11} - K_{22}N_{22} - (K_{12} + K_{21})N_{12} + (K_{12} - K_{21})\mathfrak{N} + f_3 = \dot{P}_3 + \Omega_2 P_2 - \Omega_1 P_1 \\
&M_{11,1} + M_{12,2} - Q_1(1 + \epsilon_{11}) - Q_2\epsilon_{12} + 2\gamma_{13}N_{11} + 2\gamma_{23}(N_{12} + \mathfrak{N}) - M_{12}K_{13} - M_{22}K_{23} + m_1 = \dot{H}_1 - \Omega_3 H_2 - V_1 P_3 - V_3 P_1 \\
&M_{12,1} + M_{22,2} - Q_1\epsilon_{12} - Q_2(1 + \epsilon_{22}) + 2\gamma_{13}(N_{12} - \mathfrak{N}) + 2\gamma_{23}N_{22} + M_{11}K_{13} + M_{12}K_{23} + m_2 = \dot{H}_2 + \Omega_3 H_1 - V_2 P_3 - V_3 P_2
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
(2 + \epsilon_{11} + \epsilon_{22})N = {} & (N_{22} - N_{11})\epsilon_{12} + N_{12}(\epsilon_{11} - \epsilon_{22}) + M_{22}K_{21} - M_{11}K_{12} \\
& + M_{12}(K_{11} - K_{22}) - \Omega_1 H_2 + \Omega_2 H_1 - V_1 P_2 + V_2 P_1
\end{aligned} \tag{2}$$

$(\ )_{,\alpha}$ denotes the partial derivative with respect to the two coordinates, which describe the reference plane of the plate according to 2D plate theory. (Here and throughout the paper Latin indices assume 1,2,3; and Greek indices assume values 1,2). $\dot{(\ )}$ denotes the partial derivative with respect to time. V_i and Ω_i are the velocity and angular velocity measures. $\epsilon_{\alpha\beta}$ are the in-plane generalized strains, $\gamma_{\alpha 3}$ are the transverse shear generalized strains, and $K_{\alpha j}$ are the curvatures of the deformed surface. $N_{\alpha\beta}$ are generalized in-plane forces, Q_α are generalized shear forces, $M_{\alpha\beta}$ are generalized moments, P_α and H_α are the linear and angular momenta respectively. f_i and m_α are the external forces and moments. $\mathfrak{N}$ is a Lagrange multiplier to enforce symmetry of in-plane generalized strains.

While solving the above equation, the constitutive equations may be used to replace some of the variables in terms of others. The stress resultants are written in terms of the strains measures and the generalized momenta in terms of the six generalized velocities (i.e. the three velocities and three angular velocities). Thus,

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we can write the complete formulation in terms of only 18 unknowns (11 generalized strains, three velocities, three angular velocities and $\mathfrak{N}$), which would be solved using 18 equations. Such a set of equations are formed using six of the generalized strainvelocity equations complemented by the six compatibility equations, the five equations of motion and the constraint equation involving $\mathfrak{N}$.

As the first step, the plate is assumed to be homogeneous and isotropic, thus eliminating $K_{\alpha 3}$, $\mathfrak{N}$, ϕ and Ω_3. This would result in the linear dynamic equations.

2.1 Linear dynamic equations

The linear dynamic equations model is derived by removing terms involving $K_{\alpha 3}$, $\mathfrak{N}$, ϕ and Ω_3 from equations $[1-5]$. There are five equations of motion which are

$$
\begin{aligned}
&N_{11,1} + N_{12,2} + f_1 = \mu \dot{V}_1 \\
&N_{12,1} + N_{22,2} + f_2 = \mu \dot{V}_2 \\
&Q_{1,1} + Q_{2,2} + f_3 = \mu \dot{V}_3 \\
&M_{11,1} + M_{12,2} + m_1 = \mu r^2 \dot{\omega}_1 \\
&M_{12,1} + M_{22,2} + m_2 = \mu r^2 \dot{\omega}_2
\end{aligned} \tag{3}
$$

Further, we have the strain-velocity relations

$$
\begin{aligned}
&\dot{\epsilon_{11}} - V_{1,1} = 0 && \dot{\epsilon_{12}} - V_{2,1} + \Omega_3 = 0 \\
&\dot{\epsilon_{22}} - V_{2,2} = 0 && \dot{\epsilon_{21}} - V_{1,2} - \Omega_3 = 0 \\
&\dot{\gamma_{13}} - V_{3,1} - \Omega_1 = 0 && \dot{\gamma_{23}} - V_{3,2} - \Omega_2 = 0 \\
&\Omega_{2,1} - \dot{K_{12}} = 0 && \Omega_{2,2} - \dot{K_{22}} = 0 \\
&\Omega_{1,1} - \dot{K_{11}} = 0 && \Omega_{1,1} - \dot{K_{21}} = 0 \\
&\Omega_{3,1} - \dot{K_{13}} = 0 && \Omega_{3,2} - \dot{K_{23}} = 0 \\
&\dot{K_{12}} + \dot{K_{21}} = \Omega_{2,1} + \Omega_{1,2}
\end{aligned} \tag{4}
$$

The linear and angular momenta are expressed in terms of velocities and angular velocities as

$$
\left\{ \begin{array}{c} P \\ H \end{array} \right\} = \left[\begin{array}{cc} \mu\Delta & -\mu\tilde{\xi} \\ \mu\tilde{\xi} & I \end{array} \right] \left\{ \begin{array}{c} V \\ \Omega \end{array} \right\} \tag{5}
$$

$\mu, \tilde{\xi}, I$ are, respectively, the mass per unit length, mass center offset (a vector in the cross-section from the beam reference axis to the cross-sectional mass center), and the cross-sectional inertia matrix consisting of mass moments of inertia per unit length on the diagonals.

The sectional constitutive law relates the generalized forces (in-plane, shear and moments) are related to generalized strains using the cross-sectional stiffnesses or flexibilities.

$$
\left\{ \begin{array}{c} N \\ M \\ Q \end{array} \right\} = \left[\begin{array}{ccc} R & S & 0 \\ S^T & T & 0 \\ 0 & 0 & U \end{array} \right] \left\{ \begin{array}{c} \epsilon \\ \kappa \\ 2\gamma \end{array} \right\} \tag{6}
$$

R, S, T and U are the stiffness parameters governed by the material properties and the geometry of the section.

Usually, the constitutive laws are used to replace some variables in terms of others. Here it was decided to express the generalized strains in terms of the cross-section stress resultants, allowing easy specification of zero flexibility, and the generalized momenta in terms of generalized velocities, allowing easy specification of zero inertia. Thus, the primary variables of interest are $N_{\alpha\beta}$, $M_{\alpha\beta}$, Q_α, V_i and Ω_α.

Finally the boundary conditions need to be specified. For the rectangular plate, there will be five boundary conditions along each edge. For the sake of simplicity, the plate is considered to be clamped along one of the edges (x_1=0 edge in this case) and free along the other three edges. Thus, the assumed boundary conditions are

$$\begin{array}{llll} x_1 = 0: & V_i = 0, & \Omega_\alpha = 0 & \\ x_1 = a: & N_{1\alpha} = 0, & M_{1\alpha} = 0, & Q_1 = 0 \\ x_2 = 0: & N_{\alpha 2} = 0, & M_{\alpha 2} = 0, & Q_1 = 0 \\ x_2 = b: & N_{\alpha 2} = 0, & M_{\alpha 2} = 0, & Q_2 = 0 \end{array} \tag{7}$$

3 Finite Element Formulation

The finite element formulation is based on discretizing the plate into m elements along x_1 direction and into n elements x_2 direction respectively so that there is a totally of m x n elements. For any element (i^{th} element along x_1 and j^{th} element along x_2, denoted by ij), the solution is given by V_k^{ij}, Ω_α^{ij}, $N_{\alpha\beta}^{ij}$, $M_{\alpha\beta}^{ij}$, Q_α^{ij}. In addition to satisfying the equations of motion, the kinematic equations and the boundary conditions given above, the solution must also satisfy the continuity equations between adjacent elements along all its edges. Thus,

$$\begin{array}{rr} V_1^i(L_i, x_2, t) = V_1^{i+1}(0, x_2, t) & N_{11}^i(L_i, x_2, t) = N_{11}^{i+1}(0, x_2, t) \\ V_2^i(L_i, x_2, t) = V_2^{i+1}(0, x_2, t) & N_{12}^i(L_i, x_2, t) = N_{12}^{i+1}(0, x_2, t) \\ V_3^i(L_i, x_2, t) = V_3^{i+1}(0, x_2, t) & M_{11}^i(L_i, x_2, t) = M_{11}^{i+1}(0, x_2, t) \\ \Omega_1^i(L_i, x_2, t) = \Omega_1^{i+1}(0, x_2, t) & M_{12}^i(L_i, x_2, t) = M_{12}^{i+1}(0, x_2, t) \\ \Omega_2^i(L_i, x_2, t) = \Omega_2^{i+1}(0, x_2, t) & Q_1^i(L_i, x_2, t) = Q_1^{i+1}(0, x_2, t) \end{array} \tag{8}$$

$$\begin{array}{rr} V_1^j(x_1, L_j, t) = V_1^{j+1}(x_1, 0, t) & N_{12}^j(x_1, L_j, t) = N_{12}^{j+1}(x_1, 0, t) \\ V_2^j(x_1, L_j, t) = V_2^{j+1}(x_1, 0, t) & N_{22}^j(x_1, L_j, t) = N_{22}^{j+1}(x_1, 0, t) \\ V_3^j(x_1, L_j, t) = V_3^{j+1}(x_1, 0, t) & M_{12}^j(x_1, L_j, t) = M_{12}^{j+1}(x_1, 0, t) \\ \Omega_1^j(x_1, L_j, t) = \Omega_1^{j+1}(x_1, 0, t) & M_{22}^j(x_1, L_j, t) = M_{22}^{j+1}(x_1, 0, t) \\ \Omega_2^j(x_1, L_j, t) = \Omega_2^{j+1}(x_1, 0, t) & Q_2^j(x_1, L_j, t) = Q_2^{j+1}(x_1, 0, t) \end{array} \tag{9}$$

The weighting functions are then introduced into the equations of motion, kinematic equations and the boundary conditions in a way similar to Patil and Hodges (2011):

$$\begin{aligned} \int\!\!\int [\delta V_1(N_{11,1} + N_{12,2} + f_1 - \mu\dot{V}_1) + \delta V_2(N_{12,1} + N_{22,2} + f_2 - \mu\dot{V}_2) + \delta V_3(Q_{1,1} + Q_{2,2} + f_3 - \mu\dot{V}_3) \\ + \delta\Omega_1(M_{11,1} + M_{12,2} - Q_1 + m_1 - \mu r^2\dot{\Omega}_1) + \delta\Omega_2(M_{12,1} + M_{22,2} - Q_2 + m_2 - \mu r^2\dot{\Omega}_2) + \delta N_{11}(\dot{\epsilon}_{11} - V_{1,1}) \\ + \delta N_{22}(\dot{\epsilon}_{22} - V_{2,2}) + \delta N_{12}(\dot{\epsilon}_{12} - V_{1,2} - V_{2,1}) + \delta M_{11}(\dot{K}_{11} - \Omega_{1,1}) + \delta M_{22}(\dot{K}_{22} - \Omega_{2,2}) \\ + \delta M_{12}(\dot{K}_{12} - \Omega_{1,2} - \Omega_{2,1}) + \delta Q_1(2\dot{\gamma}_{13} - V_{3,1} - \Omega_1) + \delta Q_2(2\dot{\gamma}_{23} - V_{3,2} - \Omega_2)]\, dx_2\, dx_1 \end{aligned} \tag{10}$$

Finally, each of the 13 variables in the equations is expanded in terms of a trial function. The values of the variables are assumed to be a function of the nodal values. Let there be $m \times n$ elements (i = 1, 2,. . . ,

m; $j = 1, 2, \ldots, n$) and p nodes ($k = 1, 2, \ldots, p$) within each element and $\mathfrak{F}$ be a shape function. The variables now take the form

$$\begin{gathered}
V_1^i(x^i,x^j,t) = \mathfrak{F}^k(x^i,x^j)v_1^{k,i}(t) \qquad V_2^i(x^i,x^j,t) = \mathfrak{F}^k(x^i,x^j)v_2^{k,i}(t) \qquad V_3^i(x^i,x^j,t) = \mathfrak{F}^k(x^i,x^j)v_3^{k,i}(t) \\
\Omega_1^i(x^i,x^j,t) = \mathfrak{F}^k(x^i,x^j)\omega_1^{k,i}(t) \qquad \Omega_2^i(x^i,x^j,t) = \mathfrak{F}^k(x^i,x^j)\omega_2^{k,i}(t) \\
N_{11}^i(x^i,x^j,t) = \mathfrak{F}^k(x^i,x^j)n_{11}^{k,i}(t) \qquad N_{12}^i(x^i,x^j,t) = \mathfrak{F}^k(x^i,x^j)n_{12}^{k,i}(t) \qquad N_{22}^i(x^i,x^j,t) = \mathfrak{F}^k(x^i,x^j)n_{22}^{k,i}(t) \\
M_{11}^i(x^i,x^j,t) = \mathfrak{F}^k(x^i,x^j)m_{11}^{k,i}(t) \qquad M_{12}^i(x^i,x^j,t) = \mathfrak{F}^k(x^i,x^j)m_{12}^{k,i}(t) \qquad M_{22}^i(x^i,x^j,t) = \mathfrak{F}^k(x^i,x^j)m_{22}^{k,i}(t) \\
Q_1^i(x^i,x^j,t) = \mathfrak{F}^k(x^i,x^j)q_1^{k,i}(t) \qquad Q_2^i(x^i,x^j,t) = \mathfrak{F}^k(x^i,x^j)q_2^{k,i}(t)
\end{gathered} \tag{11}$$

Thus, the problem reduces to a set of linear algebraic equations of the form

$$\left[\ A\ \right]_{kji}\left\{\ X\ \right\}_{ji} = \left[\ B\ \right]_{kji}\left\{\ \dot{X}\ \right\}_{ji} \tag{12}$$

$$\left[\ A\ \right] = \begin{bmatrix}
\frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x_1} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x_2} & 0 & 1
\end{bmatrix}$$

$$\left[\ B\ \right] = \begin{bmatrix}
\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu r^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu r^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & R & S & 0 \\
0 & 0 & 0 & 0 & 0 & S^T & T & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & U
\end{bmatrix}$$

$$\left\{\ X\ \right\} = \left\{\ V_1 \quad V_2 \quad V_3 \quad \Omega_1 \quad \Omega_2 \quad N_{11} \quad N_{22} \quad N_{12} \quad M_{11} \quad M_{22} \quad M_{12} \quad Q_1 \quad Q_2\ \right\}^T \tag{13}$$

$[A]$, $[B]$ and $\{X\}$ are applied to every element ranging from i=1,2...m and j=1,2...n.

4 Results

The equations were solved using the variable-order FEM for a simple cantilevered plate, fixed along the $x_2 = 0$ edge with the other edges free.

Table 1: Plate Properties

Dimensions	1×1×0.01 m
Young's Modulus	70 GPa
Material density	2700 kg/m^3
Poisson's ratio	0.3

The equations help us to study the bending, stretching and twisting frequencies of a plate. The properties of the plate are given in Table 1, and the results in Table 2. The results for the bending frequencies are compared with those from ABAQUS.

Table 2: Plate Structural Frequencies

Mode	ABAQUS	1×1 elements	2×2 elements	3×3 elements
Bending	8.5209	14.087	13.6618	13.6375
Twisting	–	695.6117	664.4739	613.1459
Stretching	–	5091.7507	5525.2714	5663.2779

Because of the differences in the results, work is being carried out in identifying the reasons and also checking out the alternate Galerkin approach.

5 Conclusions

A finite element solution technique,based on a geometrically-exact, fully intrinsic equations is presented and applied to an homogeneous, isotropic cantilevered plate. Right now, the reasons for the deviation of the results compared to the exact solution are being investigated. Future work would involve including the non-linearities and aeroelastic effects and extending the equations to study the dynamics of a flapping wing.

References

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